BORGES AND MATHEMATICS

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In these essays, Guillermo Martínez illuminates the esthetic relationship between mathematics and literature by identifying mathematical elements in the works of Borges and demonstrating the profound articulation of abstraction and logical structuring in his stories, his style, and in his artistic beliefs.

With the solid foundation that his dual role as story-teller and mathematician offers, and with a good sense of humor, Martínez guides us through such fundamental texts as "The Aleph," "Death and the Compass," and "The Library of Babel," discussing Borges's favorite paradoxes in detail: infinities in which the whole is not necessarily greater than its parts, the universe as a sphere with center at any point, and books with infinitely many pages.

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In his prologue to the Spanish-language edition of *Mathematics and the Imagination,* by Kasner and Newman, Borges says that mathematics, like music, can dispense with the universe. I want to thank you for dispensing with the universe—and with Argentina—in order to be here to listen to this talk this afternoon.

The angle, the slope, and the interpretation
Thomas Mann and twelve-tone music
The game of interpretation as a balancing act

Whenever one chooses an angle or a theme—and here we have chosen Borges and mathematics—that choice somehow distorts the phenomenon being studied. This is something well known to the physicists, of course. It also occurs when a critic tries to tackle a body of work from a particular angle: very soon he finds himself caught in the quicksands of interpretation. We must keep in mind that the game of interpretation is a balancing act that can become upset by too much or too little. If we take a very specialized and purely mathematical approach to Borges's writings we run the risk of remaining *above* the text. Here "above" really means "outside": we force the text to imply things that the author neither said nor intended—an error of erudition. On the other hand, if we do not completely understand the mathematical elements that are present in Borges's work, we run the risk of remaining *below* the text. Therefore I am going to attempt an exercise of equilibrium. I know that here in this room, there are people who know a lot of mathematics, but I am going to speak to those who only know how to count to ten. This is my personal challenge: everything that I say ought to be understandable by anyone who can count to ten.

There is a second, more delicate question, to which Thomas Mann referred when he was obliged to insert a note at the end of *Doktor Faustus* recognizing Schönberg's intellectual authorship of the theory of twelve-tone music. Thomas Mann did so reluctantly because he thought that this musical theory had been transmuted into something else, something that could be molded literarily by him "in an ideal context, in order to associate it with a fictitious person" (his composer, Adrian Leverkühn). In the same way, the elements of mathematics that appear in the work of Borges are also molded and transmuted into "something else," within literature, and we will try to recognize these elements without separating them from their context of literary intentions. For example, Borges begins his essay "Avatars of the Tortoise" by saying,

> There is one concept that corrupts and deranges the others. I speak not of Evil, whose limited domain is Ethics; I refer to the Infinite.

Here the playful yet sharp linking of the Infinite with Evil immediately removes infinity from the serene world of mathematics and sheds slightly menacing light on the elegant and almost technical discussion that follows. And when Borges goes on to say that the "numerous Hydra" is a foreshadowing or an emblem of geometrical progression, he is again playing the game of projecting monstrosity and "convenient horror" onto a precise mathematical concept.

What Borges knew of mathematics
Taking precautions with his library
Truth in mathematics and literature

How much mathematics did Borges know? He says in that same essay: "five or seven years of metaphysical, theological, and mathematical training would prepare me (perhaps) for properly planning a history of the infinite." The phrase is sufficiently ambiguous that it is hard to tell whether he really dedicated that amount of time to such studies or if it was only a plan for the future. But it is clear that Borges knew at least the topics contained in *Mathematics and the Imagination*, and these topics are more than enough. This book contains a good sampling of what can be learned in a first course in algebra and analysis at a university. Such classes cover the logical paradoxes, the question of the diverse orders of infinity, some basic problems in topology, and the theory of probability. In his prologue to that book, Borges noted in passing that, according to Bertrand Russell, all of mathematics is perhaps nothing more than a vast tautology. With this observation Borges showed that he was also aware of what at least in those days was a crucial, controversial, and keenly debated topic in the foundations of mathematics: the question of *what is true* versus *what is demonstrable.*

In their day-to-day work scrutinizing the universe of forms and numbers, mathematicians come across certain connections and patterns again and again, certain relationships that recur and that are always verified. By training and habit they are accustomed to thinking that if these relationships and patterns always hold true, then it must be for some discoverable reason. They believe that the universe of forms and numbers is arranged according to some external, Platonic order, and that this order ought to be deciphered. When they find the deep, and usually hidden, explanation, they exhibit it in what is called a *demonstration* or *proof.*

Thus there are two moments in mathematics, as in art: a moment that we can call illumination or inspiration—a solitary and even "elitist" moment in which the mathematician glimpses, in an elusive Platonic world, a result that he considers to be true; and a second, let's say, democratic, moment, in which he has to convince his community of peers of its truth. In exactly the same way, an artist will have fragments of a vision and then at a later time execute that vision in the writing of a book, the painting of a picture, or some other creative activity. In that sense, the creative processes are very similar. What is the difference? That in mathematics there are formal protocols under which the truth that the mathematician wishes to communicate can be demonstrated step by step from principles and "ground rules" that all mathematicians agree on. The demonstration of the value of an esthetic work is not so straightforward, however. An esthetic work is always subject to criteria of authority, to fashion, to culture, and to the personal and ultimate criterion—often perfectly capricious—of taste.

Mathematicians believed for centuries that in their domain these two concepts—truth and demonstrability—were basically equivalent: that if something were true then the reasoning behind it could be shown with a logical demonstration, a proof. On the other hand, judges in a court of law, for example, have always known that truth is not the same as demonstrability. Let's suppose that there has been a crime committed in a locked room with only two possible suspects. Both of the suspects know the whole truth about the crime: *I did it* or *I didn't do it.* There is a fact of the matter and they know what it is, but Justice can only come to this truth through indirect means: digital footprints, cigarette butts, and alibi-checking. Often, the justice system can prove neither the guilt of one nor the innocence of the other. Something similar occurs in archeology, where the notion of truth is provisional in nature: the ultimate truth remains out of range, as an unobtainable limit, being the unceasing compilation of the bones of the demonstrable.

Thus we see that in fields other than mathematics, truth does not necessarily coincide with demonstrability. Bertrand Russell perhaps worked hardest at trying to prove that *within mathematics* the two terms do indeed coincide; that mathematics is nothing more than a "vast tautology." This is also related to *Hilbert's Program,* the sweeping attempt by mathematicians to guarantee that any statement that can be proved true, by any means whatsoever, can also be demonstrated *a posteriori* using a computational algorithm that can corroborate this truth in a mechanical way, without the aid of intelligence. The goal of Hilbert's Program was basically to reduce mathematics to those statements that can be proved by a computer.

A dramatic result by Kurt Gödel, in the 1930s—his famous *Incompleteness Theorem*—showed that this is not the way things are, and that mathematics resembles criminology in this respect: there are statements that are true but which remain, nevertheless, outside of the scope of formal theories. More precisely, in any formal theory that is complex enough to develop arithmetic, there are statements that the theory itself can neither affirm nor deny: they remain suspects for whom the theory can demonstrate neither guilt nor innocence. What I want to point out is that Borges envisioned the origin of this discussion. (It does not appear that he was aware of its outcome, however.)

Elements of mathematics in the works of Borges

There are various elements of mathematics throughout Borges's works. The natural and obvious approach to this issue is to track down all of the mathematical footprints in his writings. That has been nicely done in several of the essays in the book *Borges and Science* (Eudeba). This book contains essays on Borges and mathematics, on Borges and scientific investigation, on the subject of memory, on Borges and physics. My favorite section was "Borges and Biology." After some evasiveness, the author of that section writes, almost apologetically, that after reading the complete works of Borges he is forced to conclude that there is no connection between Borges and biology. None! The man was terrified to have discovered something in this world—biology—that Borges had not touched.

Mathematical elements, however, are abundant in Borges. I am going to take advantage of my position as a writer and try to do something somewhat different: my aim is to connect the mathematical elements with stylistic elements in Borges. I am trying an approach that is stylistic rather than thematic. Some of the stories and essays where mathematical ideas loom most conspicuously are these: "The Disk," "The Book of Sand," "The Library of Babel," "The Lottery of Babylon," "On Rigor in Science," "An Examination of the Work of Herbert Quain," and "*Argumentum ornithologicum*"; the essays "The Perpetual Race of Achilles and the Tortoise" together with "Avatars of the Tortoise," "The Analytical Language of John Wilkins," "The

Doctrine of Cycles," "Pascal" together with "Pascal's Sphere," and so on. Some of these even contain small mathematical lessons. And though the topics considered are quite diverse, I see three recurring themes. Furthermore, these three themes all come together in one story, "The Aleph." I propose that we begin our study there.

Cantor's Infinity

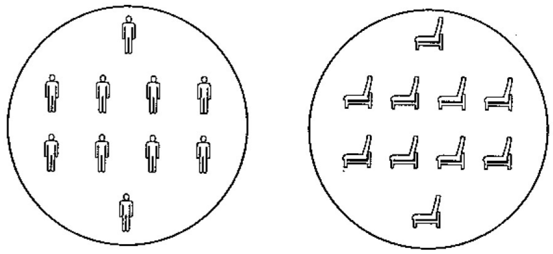
I am going to talk about these three recurring themes in reverse order. The first theme is the infinite, or, more accurately, the infinities. Toward the end of "The Aleph," Borges wrote:

> I want to add two observations: one on the nature of the Aleph, the other on its name. This, as we all know, is the name of the first letter of the alphabet of the sacred language. Its application to the disk in the story does not appear to be accidental. For the Kabbalah that letter signifies the *En Soph,* the unlimited and pure deity. It is also said that it has the form of a man who points to the heavens and to the earth, indicating that the earth below is the mirror and map of the heavens above. For *Mengenlehre* it is the symbol of the transfinite numbers, in which the whole is not necessarily greater than each of its parts.

Mengenlehre is the German word for the theory of quantities. The symbol for aleph appears thus: $\aleph$, with one arm pointing to the sky and the other pointing to the earth. It is the symbol of the transfinite numbers, in which, as Borges says, *the whole is not necessarily greater than each of its parts,* a notion that contradicts the Aristotelian postulate that a whole must be greater than any its parts. This is one of the mathematical concepts that really fascinated Borges. I would like to give a short explanation of how this idea of infinity arises in mathematics.

Until 1870, when Cantor began his work on the "theory of aggregates," mathematicians used another symbol for infinity, a sideways figure eight: ∞. They did not consider the possibility that there might be different varieties of infinity. How did Cantor arrive at his idea of infinity, which gives rise to this first paradox?

In order to understand Cantor's idea, we have to remember what it means to count. The process of counting can be thought about in two ways. Let's suppose that we have one collection of ten people—that is our finite number, ten—and a second collection of ten chairs. I might very well say that I know that there are as many people as chairs because here I count ten people, and there I count ten chairs. Or in other words, I assign to the first collection a



number that I know: ten. And I assign to the second collection a number that I know: ten. And since ten is equal to ten I conclude that the collections have the same number of elements. To take another example, let's suppose that I am playing a game of cards with a three-year-old boy. The boy, like those of us here this afternoon, cannot count past ten. Let's suppose that he gives me the first card, keeps the second, gives me the third, keeps the fourth, and so on, until the entire deck is dealt out. Since he does not know how to count past ten he cannot say how many cards he has in his hand, but he can still say *something.* Indeed, he can be reasonably certain that *he and I have the same number of cards.* This he knows even though he does not know what this number is.

In the example of the chairs, we could also have concluded that there are as many people as chairs by having each person sit in a chair and verifying that this establishes a perfect correspondence in which no chair is left without a person, and no person without a chair. In the same way, a casual glance at a military parade will not reveal how many horsemen there are, nor how many horses, but it can still reveal something: that there are as many horsemen as horses.

I realize that this is a trivial example, but sometimes the best ideas spring from trivialities. We will now see the magic of a mathematician at work. Pay attention to what Cantor did here—it is something fundamentally simple, yet extraordinary. What he found is a concept that applies to any collection, finite or infinite, but which in a finite context is equivalent to "having the same number of elements." He noticed that "in a finite context, two collections A and B have the same number of elements if and only if there is a perfect one-to-one correspondence between them." This assertion is very easy to prove. But what happens when we jump to an infinite context? Of the two concepts of "having the same number of elements" that we have discussed for finite collections—(i) counting both collections and seeing if the count is the same, and (ii) establishing a perfect, one-to-one correspondence between the two collections—one of them no longer makes sense when we pass from the finite to the infinite. What could the first notion mean for an infinite collection if the process of counting it cannot come to an end? In the context of infinite collections the notion of counting no longer makes sense. But the second notion survives: for infinite collections it is still possible to establish perfect one-to-one correspondences, just as we did between people and chairs.

But then strange things start to happen. It turns out that there is an obvious way to establish a perfect, one-to-one correspondence between the complete set of natural numbers—the numbers we use for counting—and the subcollection containing only the even numbers. To 1 we assign 2, to 2 we assign 4, to 3 we assign 6, and so on. And here we are compelled by Cantor's definition to conclude that there are "as many" natural numbers as even numbers. On the other hand, the even numbers are also "half" of the natural numbers, in the sense that the natural numbers are obtained as the union of the even numbers and the odd numbers. Thus, there is effectively a part—the even numbers—that is as great as the whole. *There is a part that is equivalent to the whole.* This is the kind of paradox that amazed Borges: in the mathematics of the infinite, the whole is not necessarily greater than any of its parts. There are proper parts that are as great as the whole. There are parts that are equivalent to the whole.

Recursive objects

This particular property of infinity can be abstracted and applied to other situations in which a part of an object encompasses the data or information content of its entirety. We will call such objects *recursive* objects. Borges's Aleph, the little sphere that encompasses every image in the universe, is a recursive object in this way, albeit a fictional one. When Borges says that the application of the name "Aleph" to this sphere is not accidental and immediately calls attention to the connection with this property of infinite sets—that a part can equivalent to the whole—he is inserting his conception into an environment that makes it plausible. This is the technique that he explains in his essay "Narrative Art and Magic," at the point where he discusses the narrative difficulty in making a centaur believable. Just as in the case of infinity, where a part can be equivalent to the whole, it is conceivable that there is an element of the universe that encompasses the data or information content or knowledge of everything.

There are other recursive objects that Borges played with in his works. For example, the maps in "Rigor in Science," where the map of a single province occupies an entire city, and "in whose abandoned parts, in the deserts, lived animals and beggars." And from the point of view of biology, human beings are recursive objects. A single human cell is enough to make a clone. Certain mosaics are clearly recursive objects: in particular, those in which the design inherent in the first few tiles is repeated throughout.

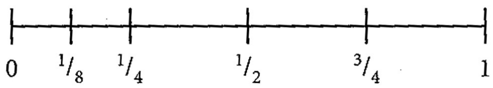
Now consider objects that have the *opposite* property. What would *anti-recurive* objects be like? They would be objects in which each part is essential and no part can be used as a replacement for the whole thing. Finite sets are examples of anti-recursive objects because no proper subset of a finite set is equivalent to the whole set. Jigsaw puzzles are also examples because, if they are good ones, no two pieces will be alike. From an existential point of view, human beings are anti-recursive. There is an intimidating phrase that is due not to Sartre but to Hegel: "Man is no more than the sum of his actions." It does not matter how flawless a man's conduct has been during each day of every year of his life: there is always time to commit some final act that contradicts, ruins, and destroys everything that has happened up to that moment. Or to take the literary turn given by Thomas Mann in *The Holy Sinner,* his book based on the life of St. Gregory: no matter how incestuous and sinful a man has been throughout his entire life, he can always confess his sins and become Pope.

Infinity and the Book of Sand

What I have said up to this point about the infinite would be enough to clarify this small fragment. I am going to extend the discussion a little further in order to explain the relationship between "The Library of Babel" and "The Book of Sand." We have just recently seen that there are "as many" natural numbers as even numbers. But what happens if we consider fractions? Fractions are very important in Borges's thinking. Why? Let us recall that fractions (also called *rational numbers*) are obtained by dividing integers. Fractions may be thought of as pairs of integers, with one integer in the numerator and another (which cannot be zero) in the denominator:

$$\tfrac{3}{5},\ \tfrac{5}{4},\ \tfrac{7}{6},\ \tfrac{7}{16},\ \ldots .$$

What property of these numbers did Borges use in his stories? *That for any two fractions there is always another one between them.* Between 0 and 1 we find $\frac{1}{2}$, between 0 and $\frac{1}{2}$ we find $\frac{1}{4}$, between 0 and $\frac{1}{4}$ we find $\frac{1}{8}$, and so on. Any number can be divided in half. Because of this, there can be no *first* number greater than zero: between any positive number and zero there is always yet another. This is exactly the property that Borges borrowed in



"The Book of Sand." Remember the moment in the story when Borges (as a character) is challenged to open the Book of Sand to its first page.

> He told me that his book was called the Book of Sand because neither the book nor sand has a beginning or end. He asked me to find the first page. I lay my left hand on the cover and opened the book, with my thumb almost touching my index finger. All was useless: there were always several pages interposed between the cover and my hand.

The front cover of the Book of Sand corresponds to zero, the back cover corresponds to the number one, and the pages in between correspond to the fractions between zero and one. Among the fractions there is no first number after zero or last number before 1. Whatever number I choose, there are always others in between. In this situation it is tempting to conjecture that the infinity of the fractions is tighter, denser, or richer than the infinity of the natural numbers. The second surprise that awaits us is that this is not the case: there are "as many" rational numbers as natural numbers. How can this be?

Since fractions come from pairs of integers, numerator over denominator, all the (positive) fractions are represented in this table:[1]

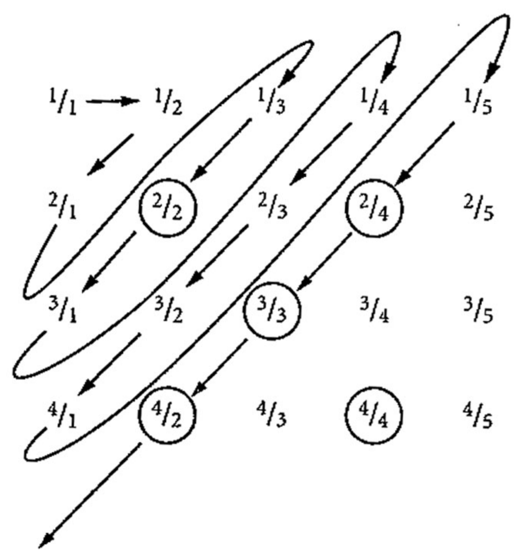
$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

The first row contains all the fractions that have numerator 1, the second row all those that have numerator 2, the third row all those that have numerator

[1] The negative fractions are not listed here, but if I can enumerate the positive fractions then it is an easy matter to enumerate the negative ones, and it is furthermore an easy matter to enumerate all of them, positive and negative, into a single list. I am speaking rather informally here; mathematicians will have to forgive me a few imprecisions.

3, and so on. Evidently, in tabulating the fractions in this way some will be repeated; for example $\frac{3}{3}$ is the same as $\frac{2}{2}$ or $\frac{1}{1}$. This repetition is not important, however: if I can enumerate the fractions in a list that includes repetitions then I can enumerate them without repetitions. What I want you to notice, what I want you to be convinced of, is that each positive fraction is located somewhere among the infinite rows and columns of this infinite table.

To show that there are "as many" positive fractions as natural numbers, it it is enough to be able to assign a distinct natural number to each element of this table. We will do this by enumerating the elements of the table in such a way that we are sure that none are left out. How can we do this? Clearly, starting with the first row and trying to enumerate all of its elements before listing anything else would result in failure: in performing this endless enumeration there would be no point at which any of the elements of the second row would be listed. Instead, if we are going to be sure to cover every element in the table, the enumeration or "path" through the table will have to move from row to row, progressively accumulating elements in the distinct rows. This method of enumerating the positive fractions was discovered by Georg Cantor, and we know it as *Cantor's Diagonal Argument.* The enumeration proceeds as follows:



To the fraction $\frac{1}{1}$ we assign the number 1.

To the fraction $\frac{1}{2}$ we assign the number 2.

To the fraction $\frac{2}{1}$ we assign the number 3.

To the fraction $\frac{1}{3}$ we assign the number 4.

We skip $\frac{2}{2}$ because it is already counted ($\frac{1}{1} = \frac{2}{2}$).

To the fraction $\frac{3}{1}$ we assign the number 5.

To the fraction $\frac{1}{4}$ we assign the number 6.

And so on.

The enumeration advances by ever longer diagonals, eventually visiting each element of each row and column of the table. As we move forward along this path we ensure that no positive fraction is missed, and if any fractions are repeated, such as $\frac{3}{3}$ or $\frac{2}{4}$, we simply skip over them. And what does this show? It shows that even though the infinite set of positive fractions does seem to have a much denser and more complicated arrangement, there are no more positive fractions than there are natural numbers.

Moreover, this enumeration gives a *consecutive* ordering to the positive fractions. This ordering is of course different from the way that the fractions lie along the number line, but it might provide an explanation for the unusual page-numbering in the Book of Sand. (This is something that Borges might not have known.) The page numbering seems mysterious to the Borges character in the story, but in principle there is no mystery. There is no contradiction between the fact that for any two leaves of the Book of Sand there is always another between them, and that each page can be assigned a unique page number: the same skillful bookbinder who could stitch those infinitely many pages into the Book of Sand could perfectly well number each page while doing so.

Infinity and the Library of Babel

Mathematicians like to take a good idea or argument, exploit it in a given setting, and then reuse it in other applications wherever possible. Borges liked to do this as well. Now that I have elucidated the idea of Cantor's Diagonal Argument I cannot resist applying it once more, on another recurring theme in Borges. This is the theme or subject of languages, as presented, for example, in "The Library of Babel."

Let's think for a moment about the idea of "The Library of Babel." The books in this Library are not necessarily intelligible. Instead, the Library is a vast collection whose fundamental principal is that "in order for a book to exist, it need only be possible." Borges fixed an alphabet of twenty-five symbols, but in order to give ourselves even more leeway, we'll imagine books written in every possible language and compile a single, universal alphabet which is the union of all symbols of all existing alphabets. We start with the twenty-five orthographic symbols that Borges mentioned in his story, so that all the books of the Library of Babel are also on our bookshelves. We continue with the twenty-seven letters of the Castillian alphabet, plus the five accented vowels as new letters. We follow those, for example, with the symbols of the Cyrillic alphabet, and then collect the German ö and the rest of the various different symbols that each language has. In this way the basic alphabet grows and grows. To give ourselves room for future growth we may suppose directly that the symbols of our alphabet are the natural numbers. Then there is always space available for the addition of new alphabets, for new symbols such as @, or for the symbols of any extraterrestrial languages that we might come across some day. The numbers from 1 to 25 correspond to the orthographic symbols of the books of the Library of Babel, the number 26 is *A,* 27 is *B,* the number 526 might be a Chinese ideogram, and so on.

Recall that in "The Library of Babel" Borges limited the number of pages in each book to four hundred and ten. We now ask: what kind of infinity corresponds to the collection of all the various books that can be written in our universal alphabet, if we admit words of *any* finite length and allow books to be of *any* finite number of pages?

Cantor's Diagonal Argument can be used to show that this collection of books *is enumerable as well.* The idea is to display all the books that consist of a single page in the first row, all the two-page books in the second row, all the three-page books in the third row, and so on. We then enumerate the books by following Cantor's diagonal path. Since every book in the Library of Babel is also included somewhere on our bookshelves, we conclude that the collection of books in the Library of Babel must also be enumerable.

How is this important in understanding Borges's story? In a note at the end of the story, Borges wrote that a lady-friend of his had observed that the entire construction of the Library of Babel was superfluous or excessive (he used the word *useless*) because all the books of the Library of Babel could fit into a *single* volume of infinitely many, infinitely thin, pages—"a silken vademecum in which each page unfolds into other pages." The book formed by piecing together all the various books of the Library of Babel into a single volume, one after another, would not be longer than Cantor's diagonal path.

I admit that this is a very mathematical way of looking at things. "The Library of Babel," is meaningful on many levels, and I am not saying that this work of literature reduces to mere mathematics. But at the end of the story Borges arrived at the idea that all of the books can be united in a single, infinite volume. This closing footnote contains the germ of the idea that foreshadows and culminates in "The Book of Sand." I want to draw attention to this way of thinking about Borges's stories and essays in order to abstract a key idea that is repeated or duplicated elsewhere. It is our first example of a literary "operation" that is reminiscent of mathematical methods. We will study this topic more thoroughly later.

The sphere with center everywhere and circumference nowhere

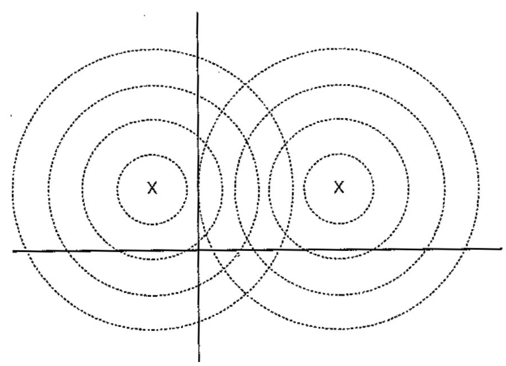
We now consider the second element of mathematics in "The Aleph." It shows up when Borges is about to describe the Aleph, and wonders "how to convey to others the infinite Aleph, that my fearful memory scarcely embraces?"

I have something more to say about the symbol for aleph. The figure of a man with one arm touching the earth and the other pointing to the sky seems particularly fitting because, in a way, the operation of counting is the human attempt at attaining infinity. That is to say, a human being cannot, in his finite life—in his "*vidita,*" as Bioy Casares would say—effectively count all the numbers. But he has a way of generating them in thought, and in this way can attain numbers as large as necessary. From the ten digits of decimal notation he can reach numbers as large as he likes. However bound to his earthly situation, he can still extend his arm to the sky. That is the objective and the difficulty of counting.

Borges wrote something similar when he asked himself "how to convey to others the infinite Aleph, that my fearful memory scarcely embraces? Mystics, in a similar hypnotic state, are lavish with emblems: to signify divinity a Persian says of a bird that it is in some way all birds; Alanus de Insulis spoke of a sphere whose center is everywhere and circumference nowhere." A little farther down he says, "the central problem—the enumeration of an infinite set—is unsolvable." Borges attempts to describe the Aleph, but it is infinite, and it is impossible to run through an infinite description in writing because writing is sequential and language is "successive." Since he cannot give a complete description of the Aleph, in its place he has to provide a sufficiently convincing idea or example, and it is his well-known enumeration of images that follows. We will have more to say on this later.

The second recurring theme is the sphere *whose center is everywhere and circumference nowhere.* This occurs in "Pascal's Sphere" and elsewhere. Borges warns his reader: "Not in vain do I recall those inconceivable analogies." It is a very precise analogy that adds plausibility to the little sphere that he describes in "The Aleph." In order to understand the geometric idea of such a sphere, something that might seem to be a play on words, we shall first ponder it in the plane, and instead of spheres we shall consider circles. Consider an ever-expanding circle: if it continues to grow indefinitely then it will eventually encompass any given point in the plane. The location of its center is not really important and it could be anywhere.

Let us make our center at any point—it is not important where—and consider larger and larger circles centered at that point. As the radii of these circles increase, the circles themselves encompass more and more of the area of the plane. In the essay, "Pascal's Sphere," at the point where



he wants to make this image a bit more precise, Borges writes, "Calogero and Mondolfo reason that Pascal intuited an infinite sphere, or rather an *infinitely expanding* one, where these words have a dynamic sense." In other words, we can replace the plane with a circle that grows and grows, because each point in the plane is eventually encompassed by such a circle. Now, in this indefinitely expanding circle, the circumference is lost at infinity. We cannot delimit any circumference. This, I think, is the idea that he is referring to. In making the jump to the infinite, the entire plane can be thought of as a circle with center at any point and circumference nowhere.

A similar construction is valid for three-dimensions: a globe that grows indefinitely will eventually encompass any given point in space. In this way, the universe can be thought of as an indefinitely expanding sphere. This, by the way, is the conception of the universe in contemporary physics: the universe was a little sphere of infinitesimal magnitude and infinitely concentrated mass that once upon a time—in the Big Bang—suddenly expanded in all directions. Why is this "inconceivable analogy" interesting? Because the Aleph is a little sphere. If the universe is viewed as a great big sphere, then the idea that every vision of the universe can be reproduced in a little sphere at the foot of the stairs in some basement is much more believable. Simply through contraction every point in the big sphere of the universe can be translated into the small sphere of the Aleph. This mathematical view is of course only one of the senses in which Borges employed this analogy. We are paying particular attention to the mathematical sense because we are "seeing everything as a cricket this morning."[2] And, as I said before, mathematics slips into Borges's writings within a context of philosophical and literary references: the idea that the universe is a sphere is connected to a whole tradition of mysticism, religion, and Kabbalah. These other connotations are explained in more detail in "Pascal's Sphere."

Russell's Paradox

The third paradox is what I call the "paradox of magnification." (The technical term in logic is *self reference,* but this has a different meaning in literature and I don't want to mix up the two concepts.) The paradox appears when Borges gives the partial enumeration of the images of the Aleph. But it also occurs in other stories, where Borges constructs worlds that are so very vast and space-filling that they end up including themselves—or even their readers—within their scopes. In "The Aleph" this can be seen here: "I saw the circulation of my dark blood, I saw the workings of love and the modification of death. I saw in the Aleph the world and in the world once more the Aleph, and in the Aleph the world. I saw my face and my guts, I saw your face I was dizzy and I cried."

Magnification, or the postulation of very vast objects, gives rise to curious paradoxes, and Borges was certainly aware of the most famous one, due to Bertrand Russell. Russell's paradox—which shook the foundations of mathematics and toppled the "naive" theory of sets—shows that one cannot postulate the existence of a set that contains all other sets; that is to say, one cannot postulate an Aleph of sets. This can be quickly explained in the following way: observe that the sets that we ordinarily think about are not

[2]From *The Cricket,* by Argentine poet Conrado Nalé Roxlo. "Is this blue sky porcelain? Is a golden cup shinbone? Or is it that in my new condition, as a cricket, I am seeing everything as a cricket this morning?"

elements of themselves. For example, the set of all natural numbers is not itself a natural number. The set of all trees is not a tree. But let's think now for a moment about the set of all *concepts.* The set of all concepts is indeed a concept. In other words, although such sets are rarer than ordinary collections of things, the idea that a set could be an element of itself seems to fit the conception of "set." Furthermore, if I postulate the set of all sets, then in order for it to be a set, it would have to be an element of itself.

Reasoning along these lines, one can distinguish between sets that are elements of themselves and others that are not. Let's consider the set of all sets that are not elements of themselves:

$$X = \{A : A \text{ is a set and } A \text{ is not an element of } A\}.$$

Then X will contain the set of all natural numbers, the set of all trees, the set of people in this room, and so on. Now we may ask ourselves: is X an element of X? The answer has to be either yes or no. If X were an element of itself then it would have to satisfy the definition written above. In other words, if X belongs to X, then X is not an element of X. But this is absurd. Does that imply then that X is not an element of itself? If X were not an element of itself, then by the definition it would have to be an element of X. In other words, if X were not an element of X, then X would have to belong to X. But this is also absurd. Here we have a set that is in Neverland, a set that neither is nor is not an element of itself.

When as a young man Russell discovered his paradox, he wrote a letter to Gottlob Frege. Frege, a very prominent mathematical logician, was on the verge of completing the final volume of an extensive treatise on the foundations of mathematics, based solely on the theory of sets. Russell's question to Frege was this: how would Frege's logical system deal with the set of all sets that do not contain themselves? In acknowledgment of Russell's communication, Frege appended his treatise with the following pathetic words: "A scientist can hardly find himself in a more undesirable situation than to see his foundations disappear just as his work has been completed. I was put in this position by a letter from Mr. Bertrand Russell just as my work was going to press." With these short lines Russell not only demolished ten or fifteen years of Frege's work, he also provoked a profound crisis in the foundations of mathematics.

To popularize his paradox, Russell came up with the story of the local barber who shaves all the men in town who do not shave themselves. On its face, the existence of a man with this honest profession seems quite reasonable: the barber, one would say, is precisely the man who shaves the men who do not shave themselves. But now we ask the question: does the barber shave himself, or not? If he shaves himself, then he has to belong to the class of those men in town who shave themselves, and hence is not shaved by the barber (that is to say, himself). On the other hand, if he does not shave himself then he is left in the class of men who do not shave themselves, and hence must be shaved by the barber (himself). The barber is trapped in a logical limbo in which his beard grows and he can neither shave it nor neglect it.

There is another variation of this paradox, also attributed to Russell, that Borges mentioned elliptically in "The Library of Babel." At the beginning of the story a librarian is in search of the catalog of all catalogs. I propose that for next week you think about formulating the paradox in terms of catalogs. Because, what are catalogs, basically? They are books that have as their contents the titles of other books. There are catalogs that list themselves among these titles and others that do not. In this way one can arrive at the same paradox.

Why are mathematicians interested in Borges?

The three elements that we have just examined appear time and again in Borges's works, molded in literary forms in various ways. In the essay, "Cartesianism as Rhetoric (or, Why are Scientists Interested in Borges?)" in *Borges and Science*, the author, Lucila Pagliai, asks why Borges's stories and essays are so dear to scientific investigators, philosophers, and mathematicians. She comes to the conclusion that there is an *essentially essayistic matrix* in the work of Borges, especially in his mature work, and I think she has a point. Borges is a writer who procedes from a single principle—"in the beginning was the idea,"—and conceptualizes his stories as incarnations or avatars of abstractions. There are also fragments of logical arguments in many of his stories. The kind of essayistic matrix that Pagliai refers to is, undoubtedly, one of the elements of Borges's style that bear a certain similarity to scientific thought.

In a little article that I wrote on the same topic, "Borges and Three Paradoxes of Mathematics," I point out the elements of Borges's style that have affinity with the mathematical esthetic. Here is my principal thesis:[3]

[3] Another excellent essay in *Borges and Science*, "Indications," by Humberto Alagia, called my attention to the fragment of "The History of Eternity" that I cited in this passage.

> I said before that traces of mathematics abound in the work of Borges. Even in the passages that have nothing to do with mathematics, there is something in his writing, an element of style, that is particularly pleasing to the mathematical esthetic. I think that a clue to this element is expressed, inadvertantly, is this extraordinary passage from *The History of Eternity*: "I don't believe in bidding farewell to Platonism (which seems ever cold) without communicating the following observation, with the hope that it will be carried forward and further justified: *the general can be more intense than the concrete.* There is no shortage of illustrations. As a boy, summering in the north of the province of Buenos Aires, I was fascinated by the rounded plain and the men who drank mate in the kitchen. But my delight was tremendous when I found out that the plain was 'pampa' and those men were 'gauchos.' The general... trumps individual details."
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> When Borges writes, he typically accumulates examples, analogies, related stories, and variations on what he wants to tell. In this way the thrust of the story that unfolds is at once particular and general, and his passages give the impression that his particular examples are self-supporting references to universal forms. Mathematicians proceed in the same way. When they study an example, a particular case, they examine it with the hope of discovering a stronger and more general property that they can abstract into a theorem. Mathematicians like to think that Borges writes exactly as they would if faced with the challenge: with a proud Platonism, as if there existed a heaven of perfect fictions and a notion of truth for literature.

This summarizes, in some way, what I think about the articulation of mathematical thought in the style of Borges. At this point it is not much more than what mathematicians call a *claim,* a statement that is affirmed in anticipation of being proved at some later point. In the next talk I will try to establish this claim, and will discuss some of Borges's nonmathematical stories and essays in this light. I thank you for having been here today. See you next week.